Int. J. Heat Mass Transfer. Vol. 32, No. 5, pp. 913–921, 1989
Printed in Great Britain

0017–9310/89 \$3.00+0.00

Stochastic modelling for contact problems in heat conduction

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(*Received* 4 *December* 1987 *and in final form* 3 *October* 1988)

Abstract—In this analysis, an attempt is made to simulate thermal conductivity of the material in the contact area between two bodies as a random variable in the process of heat diffusion. The problem is formulated as a stochastic field in contact with two deterministic fields. The randomness is considered to vary from one specimen (two bodies in contact) to another in a statistical ensemble space Ω. The stochastic response in the contact area is presented in terms of the mean value and standard deviation of the temperature field. A perturbation scheme is employed in the formulation such that the stochastic response for the present problem with intrinsic randomness can be obtained in the same manner as that in the problem with extrinsic randomness. The implicit finite difference method is used to solve the field equations governing the deterministic components of the random temperatures. The locations possessing the maximum deviations from the expected value in the random field are investigated. In the numerical examples, both Gaussian and Gamma processes are considered as the probabilistic density functions governing the random variates.

INTRODUCTION

WHEN HEAT passes through the interface between two bodies in contact, the amount of thermal energy transferred depends on the modes of heat conduction through solid-to-solid contact, heat conduction and convection through the interstitial gas region and thermal radiation among the contact surfaces. As a global evaluation, the constriction resistance in this area can be estimated by the fraction rule weighing the respective volumes of the solid media and the air relative to that of the total. The surface configurations of the two bodies in the contact area obviously dominate the modes of thermal energy transfer. Owing to irregularity of the contact pattern and its sensitive variation with the applied pressure, the relationship between the volume fraction of each medium and the constriction resistance can hardly be deterministic in reality.

Traditionally, a zonal resistance or conductance is established for the contact area such that the heat transfer through this complicated area can be estimated according to the temperature difference across it. For bare materials, Yovanovich developed a generalized analytical model for metal paraboloids [1] and extended it later to the contact between a single spherical surface and a flat plate [2]. Under this approach, it is assumed that the surface roughness of the contact surfaces is negligible and that heat conduction and convection in the surrounding fluids are absent. The complicated heat transfer modes in the contact area are absorbed in a thermal constriction resistance parameter which is a function of the deformed geometries of the two bodies in contact. Some elasticity theories are also involved in this model to characterize the contact configuration of the solid under an exerted pressure. The contact radius for a spherical particle, for example, in contact with a flat plate has been obtained under the Hertzian condition [3] and expressed in terms of Young's modulus and Poisson's ratio of the two bodies. The effects of thermal strains on the contact configuration were also considered as early as 1966 [4].

The modelling techniques used in the problems of contact between bare materials are also extended to the estimation of the thermal contact conductance of packed beds in contact with a flat surface. Typical examples can be found in refs. [5, 6]. Peterson and Fletcher [6] combined the analytical models established by Yovanovich and Kitscha [2] for bare materials and Clausing and Chao [7] involving the effects of microscopic asperities. The resulting expression for the thermal constriction resistance in their model contains, in addition to the Hertzian contact radius and the mean harmonic conductivity, the Roess microscopic constriction alleviation factor and the total number of contact points. Experimental results for the beds comprising aluminum 2017-T4, yellow brass, stainless steel 304 and chromium alloy AISI 52100 in contact with flat stainless steel 304 were obtained and checked with their model.

It is important at this point to distinguish the resistance induced by surface roughness and that induced by surface waviness. In the analytical approach, the assumption that the contact spot has a certain shape—say a circle—actually contributes to the effect induced by the surface waviness. The characteristic dimension of the spot must be confined to the continuum level,

NOMENCLATURE

b	process parameter
c	heat capacity in the contact domain
$D(\ )$	probabilistic density function in the sample space
d	physical dimension of the contact domain
$E[T]$	mean value of T
$f(\ )$	random sample function
f_A	boundary temperature
g_1, g_2	deterministic functions in thermal conductivity
g	stochastic function defined in equation (2)
i	nodal sequence in the finite difference method
k	random thermal conductivity in the contact area
k_A, k_B	thermal conductivity for solids A and B
k_M	algebraic mean of k_A and k_B
L	length of solid media A and B
m	sequence of the time increment
N	total number of nodes
n	order of the perturbation system
p	process parameter
q	heat flux vector
$S[T]$	standard deviation of T
T	random temperature in the domain of contact
T_A, T_B	deterministic temperature in solids A and B
t	physical time
Δt	equal time increment used in the finite difference method
Var $[T]$	variance of T
$V^{(n)}$	deterministic component of $T^{(n)}$
x	space variable
Δx	equal spacing among nodes in the finite difference method.

Greek symbols

α	random number of specimen in Ω
ε	amplitude of the random fluctuation of k
κ_A, κ_B	thermal diffusivity in solids A and B
κ_M	equivalent thermal diffusivity defined from k_M
λ	parameter defined as $\Delta t/(\Delta x)^2$
ρ	density of the medium in the contact domain
Ω	statistical ensemble space.

or the analysis based on a continuum formulation does not have sufficient resolution. Doubtless, this approach provides enough information for the thermal field around a cavity.

In dealing with the effects induced by the surface roughness, a different approach is needed in order to incorporate the combined effects of all the subscale interactions among the attending media in the contact area. Since the heat transport process—say a thermal diffusion equation in the solid—is formulated in a scale at least two to three orders of magnitude larger than that of the surface roughness, one may choose to treat the subscale interactions as a disturbance to the thermal properties used in the continuum formulation. Furthermore, because the contact pattern between two bodies is sensitive to the applied apparent pressure, such a disturbance is formulated more appropriately as a random variable, especially under dynamic contact situations. Under a certain stochastic process governing the probabilistic distribution of thermal conductivity in the statistical ensemble space Ω, one wants to estimate the mean value of temperature as well as its standard deviation resulting from such a disturbance. As discussed in ref. [8], the problem formulated in this manner introduces intrinsic randomness to the system, and a special perturbation technique may need to be employed to avoid difficulties in dealing with a field equation with random coefficients.

In the present study, this approach is taken and the thermal conductivity in the contact area is formulated as a random variable. Its amplitude, due to the disturbance from subscale interactions, varies from one specimen (two bodies in contact) to another in the space Ω. Conceptually, this is also the situation for a single specimen subjected to capricious contact pressure under dynamic conditions. The temperature distribution across the contact area will be presented in terms of its mean value and standard deviation. Several typical stochastic processes for the random fluctuation are to be considered in a transient problem considered as an example.

FORMULATION OF THE PROBLEM

Consider the transition of thermal conductivity from solid A to B shown in Fig. 1. The total length and contact region between the two bodies are assumed to be $2L$ and $2d$, respectively. Without loss of generality, it is also assumed that $2L$ and $2d$ are equally shared by the two media. It should be noticed that the half contact length d here is not the characteristic dimension of the surface roughness, but rather a threshold beyond which the effects of subscale disturbance cannot be detected. Obviously, the value of d should be smaller if the contact is smoother, and larger if the contact is rougher. For $x \in [-L, -d]$ and $x \in [d, L]$, therefore, thermal conductivity has deter-

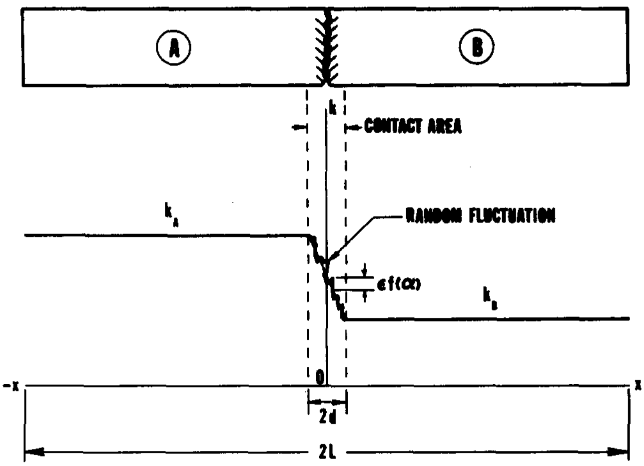

FIG. 1. Configuration of two bodies in contact and transition of the thermal conductivity.

ministic values of k_A and k_B, respectively. While in the contact region $x \in [-d, d]$, the complicated heat transport process (conduction in media A and B, and mixed conduction and free convection in the void space) is simulated by a medium with random thermal conductivity. Its amplitude ε oscillates up and down along a certain path of transition in each case, but in a series of experiments, the oscillatory pattern also randomly varies from one case to another through a stochastic sample function $f(\alpha)$. A functional form of the thermal conductivity in this domain can thus be written as

$$k(x) = k_M[(\varepsilon/k_M)g(x;\alpha)+g_2(x)] \quad (1)$$

where

$$k_M = (k_A+k_B)/2, \quad g(x;\alpha) = f(\alpha)g_1(x)$$
$$g_1(x) = x^2-d^2, \quad g_2(x) = [(k_B-k_A)/2dk_M]x+1. \quad (2)$$

The sample function $f(\alpha)$ is governed by the probabilistic density function (p.d.f.) $D(\alpha)$ under a certain stochastic process of the random variate α. It is defined in an average sense over the entire space Ω through the p.d.f.:

$$E[f(\alpha)] = \int_\Omega f(\alpha)D(\alpha)\,d\alpha,$$
$$\text{Var}[f(\alpha)] = E\{f(\alpha)-E[f(\alpha)]\}^2. \quad (3)$$

The problem formulated in this manner involves a stochastic field in contact with two deterministic fields. For $x \in [-L, -d]$ and $x \in [d, L]$, the thermal conductivities are deterministic constants, and the diffusion equation can be written as

$$T_{A,xx} = (1/\kappa_A)T_{A,t} \quad \text{for } x \in [-L, -d] \quad (4)$$

$$T_{B,xx} = (1/\kappa_B)T_{B,t} \quad \text{for } x \in [d, L] \quad (5)$$

while for $x \in [-d, d]$, due to the x-dependence of k shown in equation (1), one has

$$[k(x)T_{,x}]_{,x} = \rho c T_{,t}. \quad (6)$$

Since $k(x)$ in the contact region is a random function, equation (6) is a partial differential equation with random coefficients. As discussed in detail in ref. [8], solving this equation directly will render a situation in which the random response (say, mean value of T) of the system depends not only on the intrinsic random excitation with the same statistical order (mean value of k), but also on those with higher orders, such as covariance between k and $T_{,xx}$ and $k_{,x}$ and $T_{,x}$. Unless a statistical constitutive equation relating Cov $[k, T_{,xx}]$ and Cov $[k_{,x}, T_{,x}]$ to $E[T]$ can be established experimentally, this approach is difficult to apply, since the number of equations is always less than the number of unknowns. In this situation, thermal conductivity formulated in the form of equation (1) has its advantage in using the perturbation method suggested in the previous work. By substituting equation (1) into equation (6), one may obtain

$$[(\varepsilon/k_M)g+g_2]T_{,xx}+[(\varepsilon/k_M)g_{,x}+g_{2,x}]T_{,x} = (1/\kappa_M)T_{,t} \quad (7)$$

with κ_M being defined as $k_M/\rho c$. Equations (4), (5) and (7) constitute the governing equations for the thermal field under consideration. The initial and boundary conditions imposed on the system are assumed to be

$$T(x,0) = T_A(x,0) = T_B(x,0) = 0 \quad \text{at } t = 0 \quad (8)$$

and

$$T_A = f_A(t) \quad \text{at } x = -L$$
$$\left.\begin{array}{lll} T_A = T & \text{and} \quad T_{A,x} = T_{,x} & \text{at } x = -d \\ T_B = T & \text{and} \quad T_{B,x} = T_{,x} & \text{at } x = d \end{array}\right\} \text{(junction conditions)}$$
$$T_B = 0 \quad \text{at } x = L. \quad (9)$$

It should also be mentioned that the temperature and its gradient in the stochastic field are only defined in a mean value sense. The junction conditions in equation (9) guarantee that the statistical quantities are continuous across the interface. As far as this point is concerned, one should recall that the mean value of a deterministic quantity is equal to the quantity itself, and the variance of a deterministic quantity is equal to zero.

PERTURBATION SYSTEM OF THE GOVERNING EQUATIONS

For the contact pattern dominated by the surface roughness, the fact is recognized that the amplitude ε of the disturbance imposed on the thermal conductivity $k(x)$ (referring to equation (1)) should be small in comparison with the mean value of k_M. This suggests the possibility of using the quantity ε/k_M as a small parameter; a perturbation algorithm can thus be developed. By expanding $T(x,t;\alpha)$, $T_A(x,t;\alpha)$ and $T_B(x,t;\alpha)$ in terms of a series of ε/k_M

$$\{T(x,t;\alpha), T_A(x,t;\alpha), T_B(x,t;\alpha)\} = \sum_{n=0}^{\infty} (\varepsilon/k_M)^n \{T^{(n)}(x,t;\alpha), T_A^{(n)}(x,t;\alpha), T_B^{(n)}(x,t;\alpha)\} \tag{10}$$

and substituting equation (10) into equations (4), (5) and (7), a system of equations can be obtained for various orders of ε/k_M. The following results are obtained:

zeroth-order system

$$T^{(0)}_{A,xx} = (1/\kappa_A)T^{(0)}_{A,t} \quad T^{(0)}_{B,xx} = (1/\kappa_B)T^{(0)}_{B,t} \tag{11}$$

$$g_2 T^{(0)}_{,xx} + g_{2,x} T^{(0)}_{,x} = (1/\kappa_M)T^{(0)}_{,t} \tag{12}$$

subjected to the initial and the boundary conditions

$$T^{(0)} = T_A^{(0)} = T_B^{(0)} = 0 \quad \text{at } t = 0 \tag{13}$$

and

$$T_A^{(0)} = f_A(t) \quad \text{at } x = -L; \quad T_B^{(0)} = 0 \quad \text{at } x = L$$
$$T_A^{(0)} = T^{(0)}; \quad T^{(0)}_{A,x} = T^{(0)}_{,x} \quad \text{at } x = -d$$
$$T_B^{(0)} = T^{(0)}; \quad T^{(0)}_{B,x} = T^{(0)}_{,x} \quad \text{at } x = d \tag{14}$$

nth-order system; $n = 1, 2, 3, \ldots$ (*positive integers*)

$$T^{(n)}_{A,xx} = (1/\kappa_A)T^{(n)}_{A,t} \quad T^{(n)}_{B,xx} = (1/\kappa_B)T^{(n)}_{B,t} \tag{15}$$

$$g_2 T^{(n)}_{,xx} + g_{2,x} T^{(n)}_{,x} - (1/\kappa_M)T^{(n)}_{,t} = -g T^{(n-1)}_{,xx} - g_{,x} T^{(n-1)}_{,x} \tag{16}$$

subjected to the initial and the boundary conditions

$$T^{(n)} = T_A^{(n)} = T_B^{(n)} = 0 \quad \text{at } t = 0 \tag{17}$$

and

$$T_A^{(n)} = 0 \quad \text{at } x = -L; \quad T_B^{(n)} = 0 \quad \text{at } x = L$$
$$T_A^{(n)} = T^{(n)}; \quad T^{(n)}_{A,x} = T^{(n)}_{,x} \quad \text{at } x = -d$$
$$T_B^{(n)} = T^{(n)}; \quad T^{(n)}_{B,x} = T^{(n)}_{,x} \quad \text{at } x = d. \tag{18}$$

It is first observed that the zeroth-order system shown in equations (11)–(14) is the corresponding deterministic problem with ε being zero (0). It is then clear that the effects of subscale disturbance will gradually enter the problem as the contributions of $T^{(n)}$, $n = 1, 2, \ldots$, are summed in perturbation series (10). Another nice feature about the perturbation system can be observed from equation (16). Due to the presence of the g function and its derivative on the right-hand side of the equation, $T^{(n)}(x,t;\alpha)$, $T_A^{(n)}(x,t;\alpha)$ and $T_B^{(n)}(x,t;\alpha)$ can be expressed in the following form:

$$\{T^{(n)}(x,t;\alpha), T_A^{(n)}(x,t;\alpha), T_B^{(n)}(x,t;\alpha)\} = f^{(n)}(\alpha)\{V^{(n)}(x,t), V_A^{(n)}(x,t), V_B^{(n)}(x,t)\}, \quad n = 1, 2, \ldots \tag{19}$$

because the boundary conditions, except those for $n = 0$, are homogeneous in various orders of the perturbation system. Substituting equation (19) into equation (10), one thus obtains

$$\{T(x,t;\alpha), T_A(x,t;\alpha), T_B(x,t;\alpha)\} = \sum_{n=0}^{\infty} (\varepsilon/k_M)^n f^n(\alpha)\{V^{(n)}(x,t), V_A^{(n)}(x,t), V_B^{(n)}(x,t)\} \tag{20}$$

where the equations governing $V^{(n)}(x,t)$, $V_A^{(n)}(x,t)$ and $V_B^{(n)}(x,t)$ are the same as those shown by equations (11) and (12) for $n = 0$ and equations (15) and (16) for $n \geqslant 1$, except that g and $g_{,x}$ on the right-hand side of equation (16) are replaced by g_1 and $g_{1,x}$, respectively. It is important to notice that all the V-functions under such a formulation are deterministic. The effects of random fluctuation on the temperature field are concentrated on the factor of $f^n(\alpha)$ in the perturbation series. The solutions of T, T_A and T_B expressed in the form of equation (20) make it possible to calculate the mean value and the standard deviation of the temperature fields in a straightforward manner. For the time being, however, we will focus our attention on finding the solutions for $V^{(n)}(x,t)$, $V_A^{(n)}(x,t)$ and $V_B^{(n)}(x,t)$. Although the zeroth-order system—equations (11)–(14)—may have analytical solutions, the implicit algorithm of the finite difference method [10] will be used in this analysis. Due to the non-homogeneous term appearing in the first-order system, analytical solutions for the first and higher-order systems cease to exist, owing to the complicated integral involving the zeroth-order solution and the Jacobian of the homogeneous solutions.

The implicit finite difference algorithm is unconditionally stable and straightforward in the present problem. Supposing that the entire field is discretized into N nodes with equal spacing $\Delta x = 2L/(N-1)$, the implicit difference equations corresponding to equations (11), (12), (15) and (16) at a time instant $t = m\Delta t$, with m being a positive integer, are

$$V_J^{(n)}(i,m+1)-V_J^{(n)}(i,m) = \kappa_J\lambda[V_J^{(n)}(i-1,m+1) -2V_J^{(n)}(i,m+1)+V_J^{(n)}(i+1,m+1)] \quad \text{for } J = A, B \text{ and } n = 0,1,2,\ldots \tag{21}$$

$$V^{(0)}(i,m+1)-V^{(0)}(i,m) = \kappa_M\lambda g_2^i[V^{(0)}(i-1,m+1) -2V^{(0)}(i,m+1)+V^{(0)}(i+1,m+1)] +\kappa_M(\Delta t/\Delta x)g_{2,x}^i[V^{(0)}(i+1,m)-V^{(0)}(i,m)] \tag{22}$$

and

$$\kappa_M\lambda g_2^i[V^{(n)}(i-1,m+1)-2V^{(n)}(i,m+1) + V^{(n)}(i+1,m+1)]+\kappa_M(\Delta t/\Delta x)g_{2,x}^i[V^{(n)}(i+1,m) - V^{(n)}(i,m)]-[V^{(n)}(i,m+1)-V^{(n)}(i,m)] = -\kappa_M\lambda g_1^i[V^{(n-1)}(i-1,m+1)-2V^{(n-1)}(i,m+1) + V^{(n-1)}(i+1,m+1)]-\kappa_M(\Delta t/\Delta x)g_{1,x}^i \times[V^{(n-1)}(i+1,m)-V^{(n-1)}(i,m)], \quad n = 1,2,\ldots \tag{23}$$

where $\lambda = \Delta t/(\Delta x)^2$. In these equations, the symbol (i,m) denotes a physical quantity calculated at the ith node and the mth time instant, and superscript i denotes a function calculated at the ith node in the spatial discretization. The procedure for solving equations (21)–(23) subjected to the initial and boundary conditions (14) and (18) in various orders of the perturbation system is quite standard and will not be repeated here.

For the nodes in the contact area, the algorithm of Lagrangian interpolation is further employed to improve the accuracy in estimating the derivatives of $V^{(n-1)}(i,m)$. For a value of ε/k_M 10%, an accuracy of 0.1% can be achieved by solving the perturbation system up to the second order. For a more serious situation with, say, $\varepsilon/k_M = 0.5$, the same accuracy can be achieved by solving the system up to the tenth order. In the event that ε/k_M is even larger than this threshold, surface waviness must be considered in the formulation, but this situation is not covered in the present analysis.

STOCHASTIC RESPONSE OF THE THERMAL FIELD

The temperature field obtained from equation (20) is only meaningful in a statistical sense. For an expression with second-order approximation

$$\{T(x,t;\alpha), T_A(x,t;\alpha), T_B(x,t;\alpha)\} \simeq \{V^{(0)}(x,t), V_A^{(0)}(x,t), V_B^{(0)}(x,t)\} +(\varepsilon/k_M)f(\alpha)\{V^{(1)}(x,t), V_A^{(1)}(x,t), V_B^{(1)}(x,t)\} +(\varepsilon/k_M)^2f^2(\alpha)\{V^{(2)}(x,t), V_A^{(2)}(x,t), V_B^{(2)}(x,t)\} \tag{24}$$

the mean value and variance of $T(x,t;\alpha)$ can be obtained as [8]

$$E[T(x,t)] = \sum_{n=0}^{2}(\varepsilon/k_M)^n E[f^n(\alpha)]V^{(n)}(x,t)$$

$$\mathrm{Var}[T(x,t)] = (\varepsilon/k_M)^2\,\mathrm{Var}[f(\alpha)] \times\sum_{n=0}^{2}V^{(n)}(x,t)V^{(2-n)}(x,t) \tag{25}$$

and the standard deviation of the random temperature from its mean value

$$S[T(x,t)] = \sqrt{(\mathrm{Var}[T(x,t)])} \tag{26}$$

can be obtained consequently. At this point, a situation is achieved in which the statistical order of the random response is the same as that of the random excitation. This is a characteristic possessed by problems with extrinsic randomness, and calculations on the stochastic response of the system can thus be made in a straightforward manner.

The heat flux vector can be obtained from Fourier's law of heat conduction

$$q(x,t;\alpha) = -k(x;\alpha)T_{,x} \tag{27}$$

and from equation (27), one has

$$E[q(x,t)] = -k_M\sum_{n=0}^{2}(\varepsilon/k_M)^n V_{,x}^{(n)}\{E[f^n(\alpha)] +(\varepsilon/k_M)g_1(x)E[f^{n+1}(\alpha)]\} \tag{28}$$

for the mean value

$$\mathrm{Var}[q(x,t)] = -k_M(\varepsilon/k_M)^2\{E[f^2(\alpha)] -E^2[f(\alpha)]+(\varepsilon/k_M)g_1^2(x)(E[f^3(\alpha)] -E^3[f(\alpha)])\}\sum_{n=0}^{2}V_{,x}^{(n)}V_{,x}^{(2-n)} \tag{29}$$

for the variance, and

$$S[q(x,t)] = \sqrt{(\mathrm{Var}[q(x,t)])} \tag{30}$$

for the standard deviation of $q(x,t)$.

NUMERICAL EXAMPLES

The ratio ε/k_M will be taken as 0.1 in the following examples, and the perturbation system will be solved up to the second order. The sample function $f(\alpha)$ is assumed to have an exponentially decaying form in the space Ω:

$$f(\alpha) = \exp(-p\alpha) \tag{31}$$

while in the first example, the random variable α is assumed to be governed by a Gaussian distribution [9]

$$D(\alpha) = (1/b\sqrt{(2\pi)})\exp(-\alpha^2/2b^2). \tag{32}$$

The mean value for various orders of $f(\alpha)$ and the variance can be obtained as

$$E[f(\alpha)] = \exp(b^2p^2/2),\quad E[f^2(\alpha)] = \exp(2b^2p^2), \quad E[f^3(\alpha)] = \exp(9b^2p^2/2), \quad \mathrm{Var}[f(\alpha)] = \exp(2b^2p^2)-2\exp(5b^2p^2/2) +\exp(4b^2p^2). \tag{33}$$

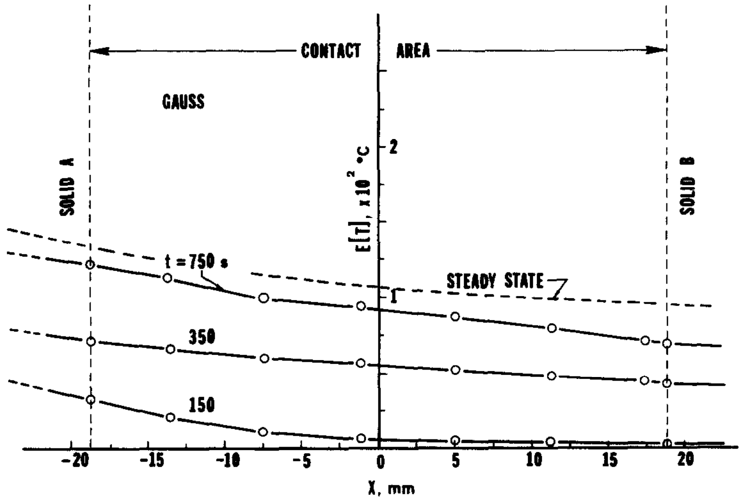

FIG. 2. Mean value of the temperature distribution in the contact area at $t = 150$, 350, 750 s, and the steady-state distribution (denoted by the dashed line). Gaussian process with $p = 0.2$ and $b = 2$.

As a second example, the same random function $f(\alpha)$ is considered, but α is assumed to be a Gamma variate. In this case, the p.d.f. can be expressed as

$$D(\alpha) = [\alpha^{b-1} \exp(-\alpha)/\Gamma(\alpha)]H(\alpha) \tag{34}$$

where $\Gamma(\ )$ and $H(\ )$ are the Gamma and Heaviside unit step functions, respectively, and the corresponding equations to equations (33) can be obtained similarly

$$E[f(\alpha)] = (1+p)^{-b}, \quad E[f^2(\alpha)] = (1+2p)^{-b},$$
$$E[f^3(\alpha)] = (1+3p)^{-b}$$
$$\mathrm{Var}\,[f(\alpha)] = [(p+1)^{2b} - (2p+1)^b]/[(2p+1)^b(p+1)^{2b}]. \tag{35}$$

For the physical and geometrical parameters involved in the system, the contact between aluminum and steel is considered in a domain of approximately 5.86% of the total length

$$L = 0.32\,\mathrm{m}, \quad d = 1.875 \times 10^{-2}\,\mathrm{m}$$
$$f_A = 250°\mathrm{C}, \quad \kappa_A = 8 \times 10^{-5}\,\mathrm{m\,s^{-1}\,°C^{-1}}$$
$$\kappa_B = 1 \times 10^{-5}\,\mathrm{m\,s^{-1}\,°C^{-1}}. \tag{36}$$

In every perturbation system with $n = 0$, 1 and 2, a total of 256 nodes (120 in each solid and 16 in the contact domain) is used in the field discretization. The time increment is selected as 50 s in the calculations. At a given time instant, based on the algorithm of

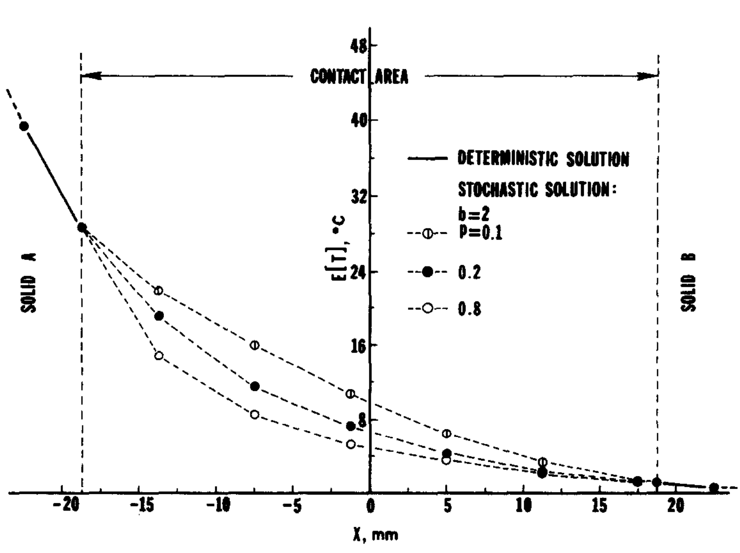

FIG. 3. The effect of process parameter p on the mean value of temperature distribution at $t = 150$ s. Gaussian process with $b = 2$.

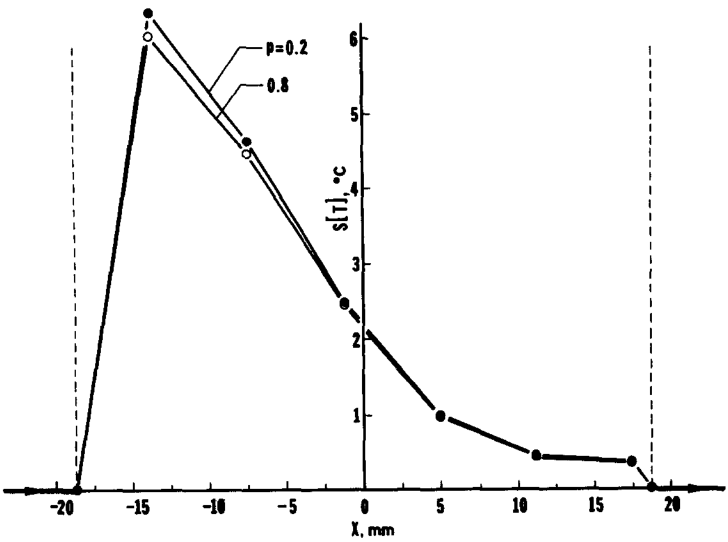

FIG. 4. Standard deviation of the temperature distribution at $t = 150$ s. Gaussian process with $b = 2$ and $p = 0.2, 0.8$.

Lagrangian interpolation, 31 nodes in the stochastic field are incorporated in the finite difference solutions for $V^{(1)}(x, t)$ and $V^{(2)}(x, t)$ with $x \in [-d, d]$. This is to enhance the accuracy in estimating the spatial derivatives involved in the non-homogeneous terms of the first- and the second-order system; as well as those appearing in the junction boundary conditions at $x = \pm d$.

With emphasis being placed on the contact domain, Fig. 2 shows the mean value of temperature distribution at $t = 150$, 350 and 750 s. The steady-state distribution is obtained by dropping the terms containing time derivatives in equations (11), (12), (15) and (16). As time increases, the temperature distribution approaches the steady state and the numerical convergence of the present algorithm is fairly clear. The process parameters b and p in the Gaussian process are taken to be 2 and 0.2, respectively. The effect of parameter p on the distribution of $E[T(x, t)]$ is shown in Fig. 3 when the value of b is fixed at a constant 2. At a given location x, the value of $E[T(x, t)]$ increases as the value of p decreases. The difference among the distributions becomes more significant if either the value of b or the boundary temperature f_A is increased. Furthermore, in view of the symmetry of b and p appearing in equation (33), one may expect the same effect of b on $E[T(x, t)]$. The standard deviation of the temperature distribution is shown in Fig. 4. It is first noticed that a maximum deviation occurs at $x \simeq -13.4$ mm in the stochastic

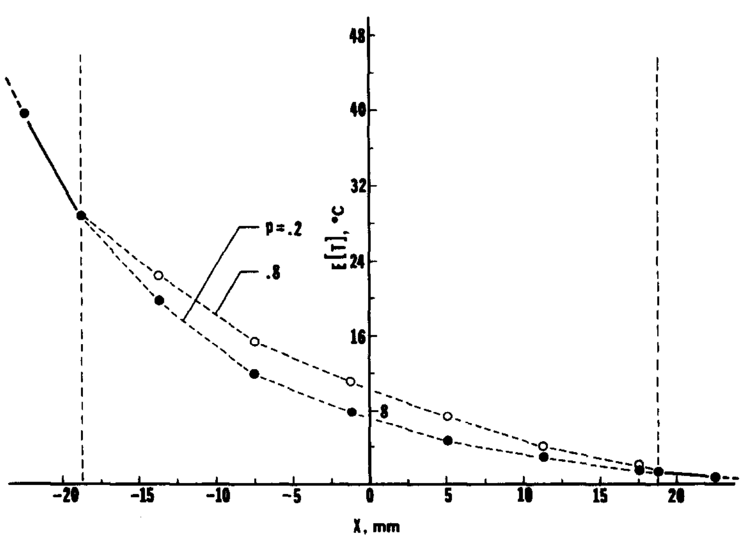

FIG. 5. The effect of process parameter p on the mean value of the temperature distribution at $t = 150$ s. Gamma process with $b = 2$.

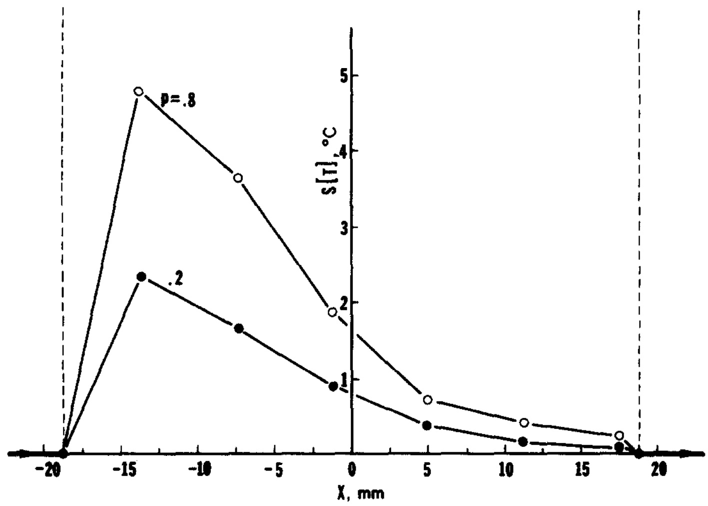

FIG. 6. Standard deviation of the temperature distribution at $t = 150$ s. Gamma process with $b = 2$ and $p = 0.2, 0.8$.

field—approximately 36% of the corresponding mean value. The standard deviation $S[T(x, t)]$, similar to $E[T(x, t)]$ but not as pronounced, decreases as the parameter p increases. Owing to the junction condition (9) imposed at the boundaries at $x = \pm d$, the values of $S[T(x, t)]$ do not vanish in the deterministic fields A and B. But the magnitude, especially in the regions far away from the boundaries, is negligible in comparison with that in the stochastic field. Figures 5 and 6 show the mean value and standard deviation of the temperature distribution under the Gamma process. Parameters b and p in this case are those defined in equation (35). With a constant value of b of 2, the effects of p on $E[T(x, t)]$ and $S[T(x, t)]$ are observed to be reversed with respect to those in the previous case. This is caused by different forms of $D(\alpha)$ between Gaussian and Gamma processes. In Fig. 6, one also observes that the differences between the maximum deviations for $p = 0.8$ and 0.2 become larger—by 22 and 12%, respectively, relative to their mean values. Again, the amount of deviation from the expected value is by no means negligible. The stochastic response for the heat flux vector can be calculated in the same fashion according to equations (28)–(30). Since it only includes spatial derivatives on deterministic functions, discussion of $q(x, t)$ will be omitted in this study.

CONCLUSION

Deterministic analysis can be adopted for an engineering system if the standard deviation is small in comparison with the corresponding mean value. In the present analysis, the amount of standard deviation has been shown to depend on the p.d.f. of a stochastic process as well as the parameters involved in it. Under the parametric values considered in the numerical examples, some cases have been proved for which stochastic analysis is necessary. The occurrence of a maximum standard deviation from the mean value reveals the importance of giving special consideration to the solid medium. It is expected that the temperature gradient in this neighborhood will also have a significant deviation, which must be considered carefully in the analysis of failure induced by the thermal field. A more complicated situation will result if the geometrical contact pattern between the two bodies is further considered as another random variable in the present formulation. In this case, the stochastic process governing the surface waviness in one specimen (two bodies in contact) will interact with that governing the random variation from one specimen to another in the space Ω. Owing to the complexity involved in this type of problem, it will be left for future communication.

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MODELISATION STOCHASTIQUE DES PROBLEMES DE CONTACT DANS LA CONDUCTION THERMIQUE

Résumé—On simule la conductivité thermique d'un matériau dans la zone de contact entre deux corps par une variable aléatoire dans le mécanisme de diffusion thermique. Le problème est formulé comme étant un champ stochastique en contact avec deux champs déterministes, avec variation d'un spécimen (deux corps en contact) à un autre dans un espace statistique Ω. La réponse stochastique dans la zone de contact est présentée en fonction de la valeur moyenne et de l'écart-type du champ de température. Un schéma de perturbation est employé dans la formulation de telle sorte que la réponse stochastique du problème présent puisse être obtenue de la même manière que dans le problème avec distribution extrinsèque. La méthode aux différences finies implicite est utilisée pour résoudre les équations de champ qui gouvernant les composantes déterministes des températures aléatoires. On étudie les régions possédant les déviations maximales par rapport à la valeur attendue dans le champ aléatoire. Dans les exemples numériques, on considère à la fois les mécanismes gaussien et gamma comme les fonctions de densité probabiliste gouvernant les variations aléatoires.

EIN STOCHASTISCHES MODELL FÜR KONTAKTPROBLEME BEI DER WÄRMELEITUNG

Zusammenfassung—Es wird versucht, die Wärmeleitfähigkeit im Kontaktbereich zwischen zwei Körpern beim Wärmediffusionsvorgang als zufallsverteilte Variable zu beschreiben. Der Vorgang wird beschrieben als ein stochastisches Feld in Berührung mit zwei deterministischen Feldern. Die Variationsmöglichkeit der Zufallsverteilung zwischen den beiden Körpern wird in einem statistischen Raum-Ensemble beschrieben. Das stochastische Verhalten im Kontaktbereich wird durch den Mittelwert und die Standardabweichung des Temperaturfeldes ausgedrückt. Ein Störungsschema wird in der Form angewandt, daß das stochastische Verhalten des vorliegenden Problems mit innerer Zufallsverteilung auf gleiche Art und Weise erhalten werden kann wie das bei externer Zufallsverteilung. Es wird die implizite Finite-Differenzen-Methode verwendet, um die Feldgleichungen der deterministischen Komponenten der zufallsverteilten Temperaturen zu lösen. Es werden die Punkte untersucht, die maximale Abweichungen gegenüber den zu erwartenden Werten des zufallsverteilten Feldes aufweisen. In den numerischen Beispielen werden sowohl Gauß- als auch Gamma-Funktionen als die wahrscheinlichsten Verteilungsfunktionen zur Beschreibung der Zufallsverteilung verwendet.

СТОХАСТИЧЕСКОЕ МОДЕЛИРОВАНИЕ КОНТАКТНЫХ ЗАДАЧ ТЕПЛОПРОВОДНОСТИ

Аннотация—Предпринята попытка смоделировать теплопроводность вещества в области контакта двух тел как случайную переменную в процессе диффузии тепла. Задача формулируется в форме стохастического поля, контактирующего с двумя детерминированными полями. Полагается, что случайность изменяется от одного образца (два контактирующих тела) к другому в пространстве статистического ансамбля Ω. Стохастический отклик в области контакта выражен средним значением и стандартным отклонением температурного поля. Используется метод возмущений, позволяющий получить стохастический отклик для данной задачи с внутренним источником случайности способом, аналогичным применяемому в задаче с внешним источником случайности. Уравнения поля, определяющие детерминированные компоненты случайных значений температуры, решаются неявным конечно-разностным методом. Исследуются области с максимальным отклонением от ожидаемого значения в случайном поле. Численные примеры рассматриваются на основе гауссовских и гамма распределений как функций плотности вероятности, определяющих случайные переменные.